Inhomogeneous Near-extremal Black Branes

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Abstract

It has recently been shown that there exist stable inhomogeneous neutral black strings in higher dimensional gravity. These solutions were motivated by the fact that the corresponding homogeneous solutions are unstable. We show that there exist new inhomogeneous black string and black p-brane solutions even when the corresponding translationally invariant solutions are stable. In particular, we show there exist inhomogeneous near-extremal black strings and p-branes. Some of these solutions remain inhomogeneous even when the size of the compact direction (at infinity) is very small.

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I. INTRODUCTION

Higher dimensional generalizations of black holes, such as black strings and black branes, have played an important role in string theory. The first solutions of this type were found by assuming translational symmetry along the brane [1,2]. However it was shown by Gregory and Laflamme [3,4] that many of these solutions are unstable to linearized perturbations which are inhomogeneous along the brane. Although it was originally believed that the endpoint of this instability (at least in the neutral case) was individual black holes, it was recently shown that this is incorrect. Instead, the solution must settle down to stable inhomogeneous black strings and p-branes [5].

In this paper we show that there exists a large class of new stable inhomogeneous black branes. These solutions are unrelated to the Gregory-Laflamme instability and exist even when the corresponding homogeneous solution (with the same mass and charge) is stable. The new solutions are charged and near their extremal limit. Gubser and Mitra [6] conjectured that for a black brane with a (noncompact) translational symmetry, the Gregory-Laflamme instability exists precisely when the brane is thermodynamically unstable. They demonstrated numerically that a certain class of black holes in anti-de Sitter space were thermodynamically unstable precisely when they were also unstable against linear perturbations [7]. Recently, Reall [8] provided strong support for this conjecture. In the cases we consider, the near extremal homogeneous black branes have positive specific heat, so they are expected to be stable. Nevertheless, we find new inhomogeneous solutions which are also stable. This shows that there can be more than one stable black brane configuration with the same mass and charge.

We cannot write down the new solutions explicitly. Instead we construct time symmetric initial data with an apparent horizon whose area is much larger than that of the homogeneous black brane with the same mass and charge. Under evolution, the mass can only decrease (through the emission of radiation) and the horizon area can only increase, so the final state will still have a horizon with area much larger than the corresponding homogeneous solution.

It must be a new stable black brane solution which is not translationally invariant along the brane.

To illustrate our construction of the initial data, consider the simple case of five dimensional Einstein-Maxwell theory. In addition to charged black holes with S^3 event horizons, this theory is known to have translationally invariant charged black strings. To keep the total mass and horizon area finite, let us compactify the direction along the string so its horizon has topology $S^2 \times S^1$. Since there is no force between extreme charged black holes, one can explicitly write down the solution for a one dimensional periodic array, which is equivalent to a black hole in a space with one direction compactified. We can now compare the horizon area for the black hole and black string with the same mass, charge, and length of the circle at infinity. Interestingly, in the extremal limit, the black string has zero horizon area while the area of the black hole remains nonzero. One can now produce new solutions by starting with the array of extreme black holes and adding thin neutral black strings to connect the horizons. From the compactified viewpoint, the black string wraps around the circle and has its ends stuck on the black hole. The total increase in mass can be made very small by taking the black string to be very thin, so the configuration is near extremality. There is an $S^2 \times S^1$ event horizon with area comparable to the extremal black hole. This configuration is not static and will evolve. But it cannot settle down to the homogeneous black string since that has very small horizon area near extremality. Instead, it must settle down to a new inhomogeneous near extremal black string. We will show that the homogeneous black string has positive specific heat near extremality, so these new solutions are not the result of a Gregory-Laflamme instability.

The new solutions have at least two surprising properties. First, in constructing the periodic array of extreme black holes, the charge Q and size of the circle at infinity L are independent parameters. In particular, one can put a large extreme black hole (with horizon area proportional to $Q^{3/2}$) inside a space with arbitrarily small L. The horizon remains a round sphere and is not distorted. Instead, the size of the circle grows as one comes in from infinity to accommodate the black hole. Our construction thus gives an inhomogeneous near extremal black string in a space with arbitrarily small L at infinity. It is unlikely that a similar statement is true for neutral black strings. In that case, the only known inhomogeneous solutions are the result of the Gregory-Laflamme instability which is only present if L is bigger than the Schwarzschild radius.

The second surprising property follows from the fact that extremal black holes have infinite throats. The thin black string that we add must go all the way down the throat to reach the horizon. Thus it has infinite length. This does not contradict our statement that the net increase in mass is very small, since the mass is infinitely redshifted near the horizon. However the infinite length implies that the area of the black string horizon is infinite. We will show that the black string horizon is an apparent horizon for this initial data, so we have an apparent horizon with infinite area in a space with finite total energy! Previously, the fact that extremal black holes have infinite throats did not seem to have any physical consequences since all matter thrown into the black hole reaches the horizon in finite time. However, we see that it can have a major effect when one adds the black string. On the other hand, the true event horizon will lie outside the apparent horizon and does not have to go down the throat. Its area will be finite and probably of order the area of the black hole.

One can do a similar construction for general black p-branes in string theory. We can start with a periodic array of extremal black branes if their horizon area is nonzero. Otherwise, we can start with a periodic array of near extremal solutions. One can then connect them with thin neutral black strings or even neutral black branes. We will see that the horizon area can be made larger than the corresponding "smeared" homogeneous solution, so it must evolve into a new inhomogeneous near extremal black brane.

In recent years the Bekenstein-Hawking entropy of near extremal black holes and homogeneous black branes has been understood in string theory. (For a review see [9].) It would be interesting to understand the entropy of these new inhomogeneous black branes. We will make some preliminary comments about this in section V.

This paper is organized as follows: In section II, we review and compare the black hole and black string solutions of five dimensional Einstein-Maxwell theory. In section III, we construct time symmetric initial data describing a periodic array of extreme black holes connected by a neutral black string. We also show that this initial data cannot evolve into the known translationally invariant solution. In section IV, we extend this construction to higher dimensions and argue that there must exist inhomogeneous near extremal black p-branes. Finally, section V contains some further discussion, including comments about the possible microstates associated with these solutions.

II. REVIEW OF STATIC SOLUTIONS

The simplest context to describe the new solutions is five dimensional Einstein-Maxwell theory with action

$$S = \int \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{4}F_{ab}F^{ab}\right] d^5x, \qquad a,b = 0,...,4 \tag{2.1}$$

where G is the five dimensional Newton constant, R is scalar curvature, and F is Maxwell field. This theory is known to have both electrically charged black holes and translationally invariant, electrically charged black strings. The latter can be viewed as resulting from the collapse of a cylinder of charged dust. The black holes are described by the five dimensional generalization of the Reissner-Nordstrom solution. We are mostly interested in the extremal limit which can easily be obtained as follows [10]. Let

$$ds^2 = -U^{-2}(x^i)dt^2 + U(x^i)\delta_{ij}dx^i dx^j \tag{2.2}$$

$$E_i = \alpha U^{-1}\partial_i U \tag{2.3}$$

where $\alpha = \pm(3/16\pi G)^{1/2}$. Then the Einstein-Maxwell equations reduce to just the condition that U be a harmonic function. If U is the field of a point mass,

$$U = 1 + \frac{\mu}{x_i x^i} \tag{2.4}$$

where μ is a positive constant, then the solution (2.2) describes an extremal black hole. The spatial metric has an infinite "throat" since the proper distance to $x^i = 0$ is stretched out infinitely, and near the origin the area of the three-spheres of constant radius is almost independent of the radius. In these coordinates, the event horizon is at $x^i = 0$ and has area

$$A_{\rm BH} = 2\pi^2 \mu^{3/2}. \tag{2.5}$$

The ADM mass M and charge Q are given by

$$M = \frac{3\pi}{4G}\mu, \qquad Q = \pm\sqrt{\frac{3\pi}{G}}\pi\mu, \tag{2.6}$$

respectively, where we have simply normalized the charge by $Q = \oint E_i dS^i$.

Solutions describing several extremal black holes are easily constructed by letting U have several point sources. (These are the analog of the familiar Majumdar-Papapetrou solutions in $3+1$ dimensions [11,12].) To compactify one direction, we let U be the field of a one dimensional periodic array of point masses. The resulting metric can be written [10]

$$\begin{aligned} ds_5^2 &= -U^{-2}(r,z)dt^2 + U(r,z)(dr^2 + r^2 d\Omega^2 + dz^2), \\ U(r,z) &= 1 + \frac{\pi\mu}{Lr}\frac{\sinh 2\pi\frac{r}{L}}{\cosh 2\pi\frac{r}{L} - \cos 2\pi\frac{z}{L}}, \end{aligned} \tag{2.7}$$

where the coordinate z is periodic with period L. The black hole horizon is located at $r = z = 0$, where U diverges. Expanding U near this point yields

$$U(r,z) = 1 + \frac{\mu}{r^2 + z^2} + \frac{\pi^2}{3}\frac{\mu}{L^2} + O\left(\frac{r^2}{L^2}, \frac{z^2}{L^2}\right). \tag{2.8}$$

So the geometry near the horizon reduces to that of the isolated black hole. For $r \gg L$, we have

$$U = 1 + \frac{\pi\mu}{rL} + O(e^{-2\pi r/L}) \tag{2.9}$$

Note that the inhomogeneity in the z direction falls off exponentially for large r. This is expected since, from a four dimensional viewpoint, z dependent perturbations act like massive fields. The ADM mass M and charge Q of the compactified solution are identical with that of the single black hole (2.6).

It is interesting to note that μ and L are independent parameters in this solution: One can fit an arbitrarily large charged black hole into a space with one direction compactified on an arbitrarily small circle (at infinity). This is possible since the size of the circle depends on U. It follows from (2.9) that when $r \sim L$ the proper length of the circle is of order $\mu^{1/2}$, independent of L.

To obtain the (translationally invariant) charged black string solution we can proceed as follows. Consider the following metric and two form

$$ds_5^2 = e^{-4\phi/\sqrt{3}} dz^2 + e^{2\phi/\sqrt{3}} g_{\mu\nu} dx^\mu dx^\nu, \qquad \mu, \nu = 0, ..., 3$$

$$F_2 = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{2.10}$$

where $g_{\mu\nu}$ is a four-dimensional metric and z is again a compactified extra-dimension with period L. As shown in [13], if all functions are independent of z, the action (2.1) is reduced to the following four-dimensional action

$$S = L \int \sqrt{-g} \left[\frac{1}{16\pi G} [R - 2 (\nabla\phi)^2] - \frac{1}{4} e^{-(2\phi/\sqrt{3})} F_{\mu\nu} F^{\mu\nu} \right] d^4x. \tag{2.11}$$

The electrically charged spherically symmetric black hole solution of this theory is given by [14,15]

$$\begin{aligned} ds^2 &= -\lambda^2 dt^2 + \lambda^{-2} dr^2 + f^2 d\Omega^2, \\ \lambda^2 &= \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{1/2}, \\ f &= r \left(1 - \frac{r_-}{r}\right)^{1/4}, \end{aligned} \tag{2.12}$$

where r_+ and r_- ($\leq r_+$) are free parameters. The Maxwell field F and the "dilaton field" ϕ are

$$F_2 = \pm \frac{1}{4r^2} \sqrt{\frac{3 r_+ r_-}{\pi G}} dt \wedge dr \tag{2.13}$$

and

$$e^\phi = \left(1 - \frac{r_-}{r}\right)^{\sqrt{3}/4}, \tag{2.14}$$

respectively. Substituting Eqs. (2.12) and (2.14) into Eq. (2.10), we obtain the five-dimensional black string solution:

$$ds_5^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{r_-}{r}\right)d\Omega^2 + \left(1-\frac{r_-}{r}\right)^{-1}dz^2. \quad (2.15)$$

The event horizon is at $r = r_+$ and has area

$$A = 4\pi r_+^2 L\left(1-\frac{r_-}{r_+}\right)^{1/2}. \quad (2.16)$$

There is a curvature singularity at $r = r_-$. In the extremal limit, $r_+ \to r_-$, the horizon area clearly goes to zero. By a simple coordinate transformation, the extremal solution can be put into the form (2.2), where now U is the field of a line source. The total mass $\tilde{M}$ and charge $\tilde{Q}$ is given by

$$\tilde{M} = \frac{L}{4G}(2r_+ + r_-), \qquad \tilde{Q} = \pm L\sqrt{\frac{3\pi r_+ r_-}{G}}. \quad (2.17)$$

In terms of $\tilde{Q}$ and the Hawking temperature

$$T_H = \frac{1}{4\pi r_+}\sqrt{\frac{r_+ - r_-}{r_+}} \quad (2.18)$$

we can calculate the specific heat C defined as

$$C(\tilde{Q}, T_H) = \frac{\partial M(\tilde{Q}, T_H)}{\partial T_H}. \quad (2.19)$$

In the near-extreme limit $(r_+ \to r_-)$, we obtain the asymptotic form of C:

$$C(\tilde{Q}, T_H) = \sqrt{\frac{16\pi G}{27}}\frac{|\tilde{Q}|^3}{L^2}\big(T_H + O(T_H^3)\big). \quad (2.20)$$

Since the specific heat C is clearly positive in the near-extreme case, the solution (2.15) is believed to be stable.

Equating the mass of the black hole to that of the black string in the extremal limit $r_+ = r_-$, we find $\mu = Lr_+/\pi$. It then follows from Eqs. (2.6) and (2.17) that the charges of the two systems are also equal. But we have seen that the horizon areas are very different. We will use this fact in the next section to construct initial data for a new inhomogeneous near extremal black string.

III. INITIAL DATA FOR AN INHOMOGENEOUS BLACK STRING

In this section we show that one can start with the extremal black hole solution in a space with one direction periodically identified, and add a small amount of energy so that the horizon changes from having topology S^3 to having topology $S^2 \times S^1$. The idea is to add a thin neutral black string which goes around the circle and has its ends stuck on the black hole. This configuration will not be static and will evolve. We do not know the final exact static solution, but we will construct time symmetric initial data and then argue that it must settle down to a new inhomogeneous near extremal black string. There are in fact two different ways to construct the initial data. The first is more explicit, and the second is more general but (at the moment) less rigorous. We will concentrate on the first method and comment on the second at the end of this section.

A. Conformal rescaling construction

Let us consider the following four-dimensional spatial metric:

$$\begin{aligned} ds_4^2 &= u \left[\left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 + dz^2 \right] \\ &= u \, \overline{ds^2}, \end{aligned} \tag{3.1}$$

where u is a function of (r, z, θ, ϕ). z is a periodic coordinate with period L. In the case of $u = 1$, the metric represents the spatial part of the neutral black string solution with Schwarzschild radius $r = r_0$, while the $r_0 = 0$ case reproduces a spatial slice in the extremal solution (2.2).

Assuming time symmetric initial data, the constraint equations reduce to

$$R = 8\pi G E_i E^i \tag{3.2}$$

$$\nabla_i E^i = 0, \tag{3.3}$$

∇ is the covariant derivative with respect to the metric ds_4^2. Since the scalar curvature $\bar{R}$ of the metric $\overline{ds^2}$ is equal to zero, R can be simply written in terms of u:

$$R = \frac{3}{2}\frac{(\bar{\nabla}u)^2}{u^3} - 3\frac{\bar{\nabla}^2 u}{u^2}, \tag{3.4}$$

where $\bar{\nabla}$ represents the covariant derivative operator of the metric $\overline{ds^2}$. Setting

$$E_i = \alpha u^{-1}\bar{\nabla}_i u, \tag{3.5}$$

where α is a constant, the constraint Eq. (3.3) is reduced to the Laplace equation

$$\bar{\nabla}^2 u = 0. \tag{3.6}$$

Substituting Eq. (3.5) into Eq. (3.2), we can obtain α as

$$\alpha = \pm\sqrt{\frac{3}{16\pi G}}, \tag{3.7}$$

So Eq. (3.6) is the only constraint equation for the time symmetric initial data.

Let u be the solution with a point source at $\theta = 0, r = r_0, z = 0$. This corresponds to adding an extreme black hole at the north pole of the black string. Introducing a new coordinate $r_* = \int_{r_0}^{r} dr/\sqrt{1 - r_0/r}$, we can make an analytic continuation to the region $r_* < 0$. Hence this initial data has a symmetry $r_* \leftrightarrow -r_*$. Since the initial data is time symmetric, the outgoing null expansion θ_+ for any closed three-dimensional hypersurface is given by the extrinsic curvature of the hypersurface on the initial data. This means that $r_* = 0$ $(r = r_0)$ is an apparent horizon. To estimate the area, let us consider the local geometry around the north pole located at $r_* = 0$ and $\theta = 0$. Since the spatial metric $\overline{ds^2}$ is locally flat, $\overline{ds^2}$ is approximately

$$\overline{ds^2} \sim d\tilde{x}^2 + d\tilde{y}^2 + dz^2 + dr_*^2, \tag{3.8}$$

where $(\tilde{x}, \tilde{y})$ are local orthogonal coordinates around the north pole. The approximate solution of u becomes

$$u \sim \frac{\mu}{\tilde{R}^2} \qquad \text{for} \qquad \tilde{R}^2 \equiv \tilde{x}^2 + \tilde{y}^2 + z^2 + r_*^2 \ll r_0, \tag{3.9}$$

where μ is a positive constant proportional to the charge. The area of the apparent horizon around the north pole is approximately given by

$$A \sim \int \frac{d\tilde{R}}{\tilde{R}}\bigg|_{r_*=0} \to \infty. \tag{3.10}$$

This shows that the apparent horizon ($r = r_0$) has an infinite area! Since the apparent horizon has $S^1 \times S^2$ topology, the event horizon also has $S^1 \times S^2$ topology [1].

At first glance, the area of the event horizon also seems to be infinite because the event horizon is outside the apparent horizon. This speculation is obviously wrong because outside of the apparent horizon u is finite everywhere on the hypersurface. Although the exact location of the event horizon is unknown, we can still produce a lower bound on its area. The idea is to consider a region of the space and show that all surfaces passing through this region have area larger than some constant. Our estimate will be crude, but sufficient to establish the existence of new inhomogeneous solutions.

Adding the extreme black hole at the north pole of the black string horizon produces initial data which is approximately spherically symmetric for $r \gg r_0$. It will be convenient to have initial data which is approximately spherically symmetric even for $r \sim 2r_0$. This can be obtained by distributing a large number N of extreme black holes each with charge $2\,Q/N > 0$ uniformly on the sphere $(r = r_0, z = 0)$ [2]. We now construct a region such that the effects of both the compactified extra dimension and the tiny black string are negligible. To this end, let $r_b = N r_0$. By choosing r_0 small enough we can arrange for $r_b < L/N$. Introducing a new radial coordinate $R^2 = r^2 + z^2$ and angular coordinate χ such that $r =$

[1]Strictly speaking, the topology of the apparent horizon is $S^1 \times S^2$ minus a point p. However, one can modify the horizon so that the topology becomes $S^1 \times S^2$ by covering the neighborhood of p with a very shallow cap. As easily checked, the outgoing null expansion on the cap cannot be positive since u rapidly decreases toward the outside. This produces a compact apparent horizon with arbitrarily large (but finite) area.

[2]Because our initial data has two asymptotically flat regions, half of the total electric flux reaches each one. Therefore, we put $2\,Q$ on the sphere $(r = r_0, z = 0)$ as a total charge so that the total flux for one asymptotically flat region is just Q.

$R\cos\chi$, $z = R\sin\chi$, define region B by $(R,\xi) \in [\sqrt{2}\, r_b \le R \le L/N,\ -\pi/4 \le \chi \le \pi/4]$ (see Fig. 1). Then, the approximate solution in Eq. (2.8) is valid everywhere in B. Also define region A by the almost all spherically symmetric region enclosed by the $u =$ const. surface intersecting a point ($r = r_b$, $z = r_b$) (see Fig. 1). Since the electric field normal to each $u =$ const. surface should be positive, i.e. $E_i\, n^i \sim -(\bar{\nabla}_i u)\, n^i > 0$, (where n^i is a unit outward normal vector for each $u =$ const. surface), u must be increasing as one moves in. So the following lower bound on u is obtained:

$$u|_A \ge u|_{(r,z)=(r_b,r_b)} = u_b \sim \frac{\mu}{r_b^2} \tag{3.11}$$

by Eq. (2.8) (see also Fig. 1).

Now, one can estimate a lower bound A_0 for the area of the event horizon. Fig. 1 shows typical cases for the event horizon passing through our regions A or B. In general it is possible that the horizon is more complicated and wavy. In that case the area would be greater than the lower bound of the typical cases. So, it is enough for the estimation of A_0 to consider only the typical cases. Recalling the spatial metric Eq. (3.1), let us first consider the case that the event horizon intersects the region A. Then, the area of the event horizon in this region A_0 is roughly

$$A_0 \sim \left(\int_A (\sqrt{u}r)^2 \sqrt{u}\, dz\right) \ge \left(\int_A (\sqrt{u_b}r)^2 \sqrt{u_b}\, dz\right) \ge \mu^{3/2} \left(\frac{r_0}{r_b}\right)^2 = \frac{\mu^{3/2}}{N^2}, \tag{3.12}$$

where we used $r_0/r_b = 1/N$. Note that this lower bound is independent of r_0. In the case that the event horizon intersects the region B, we can easily obtain a lower bound on A_0. Eq. (2.8) is the approximate solution for the region B, and we will assume the constant term is negligible. (If necessary, we can choose N so that $\mu/L^2 \gg 1/N^2$ to insure this.) Then the area of each $u =$ const. surface is almost $\mu^{3/2}$. So, if the orbit of the event horizon is $R = R(\chi)$,

$$\begin{aligned} A_0 &\simeq 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (uR^2)^{\frac{3}{2}} \left(1 + \left(\frac{R'}{R}\right)^2\right)^{\frac{1}{2}} \cos^2\chi\, d\chi \\ &\ge 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (uR^2)^{\frac{3}{2}} \cos^2\chi\, d\chi \sim \mu^{\frac{3}{2}}, \end{aligned} \tag{3.13}$$

where a dash denotes the derivative with respect to χ. We used Eq. (2.8) to derive the final value. Finally, suppose the event horizon passes outside of the region B, and consider the subset with $-L/N < z < L/N$. In this region u is given by (2.7). From this explicit solution one can show that $ur^2 > \mu$. So the proper area of the two-spheres in this region is at least μ. Since the length in the z direction is at least L/N (since $u > 1$ everywhere) we obtain the lower bound $A_0 > \mu L/N$. Thus, we can get the lower bound for the area of the event horizon as

$$A_0 > \min\left\{\frac{\mu^{3/2}}{N^2}, \mu\frac{L}{N}\right\}, \tag{3.14}$$

which is independent of r_0.

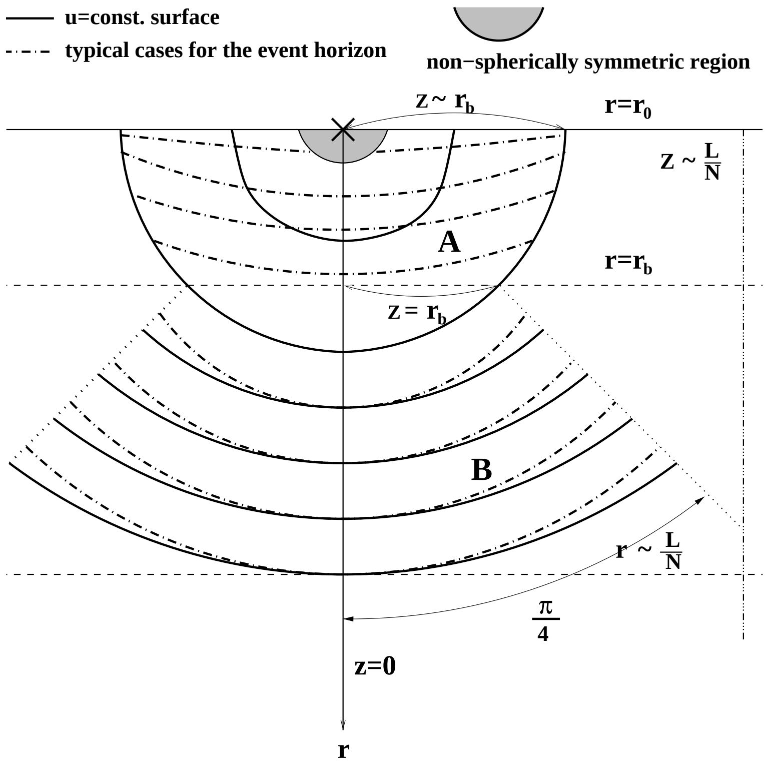

FIG. 1. Constant u surfaces and possible cases for the event horizon are depicted on the (r, z) surface. Each point represents a two-sphere. For the region B, the event horizon would be more convex than the constant u surfaces because the outgoing null expansion on those surfaces is almost zero due to the infinite throat. (The outgoing null expansion along the event horizon should be strictly positive when the spacetime is not static).

B. Evolution of initial data

The initial data constructed above must evolve since, e.g., the surface gravity is not constant over the horizon. It is large near the thin black string and small near the extremal black hole. More physically, one would expect the extremal black hole to start swallowing up the string. What will it evolve to? We will assume cosmic censorship holds, since a generic violation of cosmic censorship in five dimensions would be an even more surprising conclusion than we find below. We will also use the result of [5] that cycles on an event horizon cannot shrink to zero size. This means that if the event horizon is initially $S^2 \times S^1$, it must remain $S^2 \times S^1$. It cannot pinch off.

The ADM mass of our initial data set is easily computed from (3.1). Since u is a solution to Laplace's equation in the black string background with a point source, the asymptotic form of u can be directly related to the source by integrating the equation over all space. This means that the asymptotic form is independent of r_0 and the same as (2.9): $u \simeq 1 + \pi\mu/Lr$. The ADM mass is thus

$$M_{ADM} = M + \frac{r_0 L}{2G}. \tag{3.15}$$

where M is related to μ by (2.6). The total charge Q is also related to μ by (2.6). So the mass is above the extremal limit just by the second term which can be made very small by choosing r_0 small. Under evolution, the mass can only decrease since energy can be radiated away to infinity. Since the charge is conserved, the final state must be even closer to extremality. On the other hand, the horizon area can only increase during the evolution by the usual area theorem. Thus the final state must be a black string with horizon area larger than the initial area (3.14). But a homogeneous black string (2.15) with small excess energy $r_0 L/2G$ above extremality would have $r_+ - r_- = 4r_0$. Thus its horizon area (2.16) would be $A_0 \propto r_0^{1/2}$. By choosing r_0 sufficiently small, this can clearly be made much smaller than the initial horizon area. So the initial data cannot settle down to the homogeneous black string. It must approach a new stable, inhomogeneous near extremal black string.

We have been discussing classical solutions, but if one has to take r_0 less than the Planck scale (or string scale) the solutions would not be physically interesting. Quantum effects would certainly be important near the horizon. Fortunately, it is easy to see that this is not necessary. For our lower bound on the horizon area, $N \sim 100$ should be more than adequate. So as long as μ and L are much larger than the Planck scale, we can choose r_0 greater than the Planck scale and still deduce the existence of new solutions.

It is plausible that the final configuration will resemble a large near extremal black hole with a near extremal black string going around the circle[3]. This configuration would have low surface gravity everywhere. If one starts with this inhomogeneous black string and takes its extremal limit, it is plausible that the solution degenerates to an extremal black hole with a (singular) extremal black string going around the circle. This solution can be constructed explicitly. Recall the general form of static extremal solutions (2.2). The solution we want is obtained by letting U be the harmonic function with a line source and a point source superposed on the line. The line source produces the extremal black string, while the point source reproduces the extremal black hole. In fact, there is a large class of extremal solutions of this type, since one can specify the line density arbitrarily. It is not clear whether there are near extremal analogs of all of these extremal solutions, or just a few. If all of these extremal solutions can be obtained as the extremal limit of some black string, then the structure of black string solutions would be very rich indeed. It would depend on an arbitrary function.

C. Gluing construction

The previous construction of time symmetric initial data is very explicit, but restricted to situations where one wants to add black strings to extremal black holes. In the next section we will want to add black strings to near extremal black p-branes. To do this we need a more general construction which we describe here. This construction is less explicit (and, at the moment, less rigorous) but can be used to insert a thin black string into essentially any initial data set.

Intuitively, it seems clear that one should be able to take any smooth initial data set and insert a very small Schwarzschild black hole. This is because every curved space is locally flat, and the Schwarzschild black hole is approximately flat after a few Schwarzschild radii from the black hole. (Near the black hole there is large curvature, but the constraint

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[3]We thank S. Ross for suggesting this.

equations are satisfied.) This has indeed been proven in $3+1$ dimensions in a recent paper [16]. We would like an analogous result for black strings in $4+1$ dimensions. In other words, it should be possible to remove a neighborhood of a geodesic in an initial data set and glue in a thin black string. This possibility is supported by the fact that the argument in [16] can be easily generalized to show that one can glue small black holes in $4+1$ dimensions [17]. But if one adds a sequence of closely spaced black holes, there will be a single apparent horizon enclosing all of them which is essentially a black string.

In the case of interest here, we start with the exact solution describing a large five dimensional extremal black hole with the z direction compactified. The curvature on a static slice is everywhere small. The z axis comes up the infinite throat of the extreme black hole, goes around the circle and then back down the infinite throat. We now want to remove a neighborhood of this curve and glue in a thin black string, i.e., $4D$ Schwarzschild cross $\mathbf{R}$. The fact that the initial data contains an electric field should not be a problem. Let the electric field along the z axis be $E_z(z)$. Since the electric field is parallel to the z axis, when we glue in the black string, we can add an electric field (pointing in the $\mathbf{R}$ direction) equal to $E_z(z)$.

Although we cannot write down this initial data explicitly, we can write down a metric which describes the solution asymptotically down the throat of the extreme black hole. Before we add the black string, the asymptotic solution is just $S^3 \times \mathbf{R}$ with a constant electric field directed up the throat. So the constraint equation is just that the scalar curvature is a positive constant. Adding the black string on either side of the throat, corresponds to adding two small black holes at the poles of the large S^3. But this is exactly the geometry of a static slice of Schwarzschild de Sitter in $3+1$ dimensions. This metric has constant positive scalar curvature by virtue of the Einstein constraint equation with a positive cosmological constant. So the initial data down the throat of the extremal black hole with a black string added is just the product of a line and a static slice of $3+1$ Schwarzschild de Sitter. The electric field is still constant and directed along the line.

The resulting initial data differs from the one obtained by the conformal approach in the following respect. In the conformal approach, the size of the black string apparent horizon is rescaled by u. Thus, near the extremal black hole it becomes much larger than r_0. In fact, it is easy to see that the $r = r_0$ surface intersects the extremal black hole horizon at the equator. So the black string has grown to a size given by M! In the gluing approach this does not happen. The size of the black string stays small, $O(r_0)$, all the way down the throat.

IV. GENERALIZATION TO P-BRANES

A. Starting with extremal branes

The above construction can be applied to a variety of higher dimensional situations to deduce the existence of inhomogeneous near extremal p-branes. First we place the example we have already discussed in the context of string theory. Five dimensional Einstein-Maxwell theory arises from ten dimensional (Type II) string theory by compactifying on T^5 and considering three fundamental charges. The standard choice is D1, D5 and momentum[4]. However, due to the momentum, the ten dimensional metric has an ergoregion which complicates the construction. This can be avoided by applying a T-duality, so that the solution contains D0, D4, and fundamental string charge. If we let the charges all be different, the extremal solution has the following (string frame) metric [20]:

$$\begin{aligned} ds^2 = &- H_0^{-1/2} H_4^{-1/2} H_1^{-1} dt^2 + H_0^{1/2} H_4^{-1/2} (dy_i dy^i) \\ &+ H_0^{1/2} H_4^{1/2} (H_1^{-1} dx^2 + dr^2 + r^2 d\Omega + dz^2) \end{aligned} \tag{4.1}$$

where

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[4]Starting with this extremal solution, one can add a traveling wave along the D1 direction [18]. However the local horizon geometry remains homogeneous and is independent of the wave profile [19].

$$H_0 = 1 + \frac{\mu_0}{r^2 + z^2}, \qquad H_1 = 1 + \frac{\mu_1}{r^2 + z^2}, \qquad H_4 = 1 + \frac{\mu_4}{r^2 + z^2} \tag{4.2}$$

are three harmonic functions, and y_i are coordinates on T^4. The 4-branes are wrapped around this T^4, and the fundamental strings are wrapped in the x direction which we also assume is compact. The branes are all distributed uniformly over these compact directions. There are four directions transverse to all the branes labelled by z, r, θ, ϕ. There is an event horizon at the origin of these coordinates, $z = r = 0$, with nonzero area. Since this event horizon is extended in the x, y_i directions, the solution describes an extremal black five-brane. Its horizon has topology $S^3 \times T^5$. If we set $\mu_0 = \mu_1 = \mu_4$, then the solution (4.1) reduces to the product of (2.2) and a flat T^5.

Since H_i are just harmonic functions, we can compactify the z direction by taking a periodic array as before. The homogeneous solution with the same mass and charges can be obtained by choosing the harmonic functions to be independent of z, $\tilde{H}_i = 1 + \tilde{\mu}_i/r$. The near extremal solution is obtained by adding a factor of $(1 - r_0/r)$ to g_{tt} and $(1 - r_0/r)^{-1}$ to g_{rr} as usual. It is easy to verify that the near extremal homogeneous solution has positive specific heat, so by the Gubser-Mitra conjecture [6] it should be stable. Nevertheless, we now show that there exist near extremal solutions which are inhomogeneous in the z direction.

To begin, note that the horizon of the homogeneous solution becomes singular (zero area) in the extremal limit. This is expected since if the solution is homogeneous in the z direction also, it can be dimensionally reduced to a four dimensional black hole. But it is well known that extremal black holes in four dimensions will have nonzero area only if there are four nonzero fundamental charges [21,22]. Since we have only three nonzero charges, the four dimensional extremal black hole must have zero area.

Now, starting with the array of extremal black five branes in ten dimensions, one has several possibilities to construct initial data that will evolve into a new inhomogeneous black brane solution. The basic idea is the same as before. By adding a small amount of energy we can cause the horizon to extend in the z direction. Since the initial horizon area is much larger than the corresponding near extremal homogeneous solution, the final state must be something new.

In ten dimensions, there are neutral black p-branes which are just the product of $\mathbf{R}^p$ and the $(10-p)$-dimensional Schwarzschild solution. We can use any of the solutions with $1 \leq p \leq 6$ to connect the extremal black five-branes in the periodic array. For example, one possibility is to add a thin neutral black string in ten dimensions which goes around the z direction and ends on the black five-brane. This will produce an apparent horizon with topology $S^3 \times T^5$ with a $S^7 \times \mathbf{R}$ handle added[5]. Under evolution, the topology of the apparent horizon can change, so the final topology of the event horizon may be different. But the solution cannot settle down to the known homogeneous solution. Another possibility is to use a thin neutral black six brane to connect the extremal elements in the array. (This is just the lift of the black string we added in five dimensions in the previous section.) This will produce an apparent horizon with topology $S^2 \times T^6$, which is the same topology as the horizon in the near extremal homogeneous solution. But once again, the initial area is too large for the solution to become homogeneous. Clearly, one can do a similar construction with any of the thin neutral black p-branes with $1 \leq p \leq 6$.

One should be able to use either the gluing approach or a generalization of the conformal approach (where one rescales different parts of the spatial metric by different conformal factors) to construct suitable initial data.

B. Starting with near-extremal branes

The above solutions contained three nonzero charges in ten dimensions. If one starts with just a single charge, then the horizon area always goes to zero in the extremal limit. For example, consider a near-extremal D3-brane wrapped on a T^3 of volume V:

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[5] That is, remove two small balls from $S^3 \times T^5$. The boundary of each ball is an S^7. Now identify these two boundaries.

$$ds^2 = H_3^{-1/2}(\rho)\left[-\left(1-\frac{\rho_0^4}{\rho^4}\right)dt^2 + dy_i dy^i\right] + H_3^{1/2}(\rho)\left[\left(1-\frac{\rho_0^4}{\rho^4}\right)^{-1} d\rho^2 + \rho^2 d\Omega_5\right] \quad (4.3)$$

where $H_3(\rho) = 1 + \mu_3/\rho^4$. The horizon is at $\rho = \rho_0$ and has area $A = \pi^3 \rho_0^5 V H_3(\rho_0)^{1/2}$. This area goes to zero in the extremal limit ($\rho_0 = 0$, μ_3 fixed) so if we start with a one dimensional array of extremal three-branes and then add neutral black strings, the area of the apparent horizon will still be small and not obviously larger than the corresponding homogeneous solution. However, we can still show the existence of inhomogeneous solutions if we start with an array of near-extremal 3-branes. Such solutions are not known explicitly, but if the separation between the branes L is large compared to ρ_0, it is plausible that one can approximate the solution by starting with (4.3) and letting H be the harmonic function on flat $\mathbf{R}^6$ with a one dimensional array of point sources separated by L. Near each brane, the metric will reduce to the single brane solution (4.3), and for $r_0 = 0$, the metric reduces to the known array of extremal three branes. Let us label the direction along the array by z.

We now want to compare the horizon area of this near extremal array with that of a solution with the same mass and charge which is uniformly smeared over the z direction. That solution is easily obtained by starting with the near extremal $D4$-brane and applying T-duality along one direction of the brane:

$$ds^2 = \tilde{H}_3^{-1/2}(r)\left[-\left(1-\frac{r_0^3}{r^3}\right)dt^2 + dy_i dy^i\right] + \tilde{H}_3^{1/2}(r)\left[\left(1-\frac{r_0^3}{r^3}\right)^{-1} dr^2 + r^2 d\Omega_4 + dz^2\right] \quad (4.4)$$

where $\tilde{H}_3(r) = 1 + \tilde{\mu}_3/r^3$. One can easily check that this solution has positive specific heat and should be stable to small perturbations. It turns out [23] that the array has larger horizon area than the homogeneous solution whenever L is larger than r_0. (Unlike the previous cases where we could let L be arbitrarily small, here we require $L > r_0$.) We are now in a similar situation as before and can deduce the existence of new inhomogeneous solutions. The simplest possibility is to connect the elements of the array with a neutral black 4-brane, i.e. the product of 6D Schwarzschild and T^4. The 4-brane wraps the same T^3 as the 3-brane, plus the z direction. Since the curvature of the near extremal 3-brane is small near the horizon if the charge is large, it should be possible to construct this initial data using the gluing construction. By exactly the same arguments as in section 3, this initial data will evolve, but it cannot settle down to the homogeneous smeared 3-brane solution since that solution has a horizon which is too small. It must settle down to a new smeared black 3-brane, which is near extremal but inhomogeneous in the direction perpendicular to the branes.

Another possibility is to add a ten dimensional neutral black string to connect the elements of the array. The apparent horizon would now have the topology of $T^3 \times S^5$ with a $S^7 \times R$ handle attached. As before, the final topology of the event horizon could be different, but it could not settle down to the homogeneous solution. One could also glue in neutral black 2-branes or 3-branes.

We have started with large near extremal 3-branes. But one could equally well start with any near extremal brane such that the curvature is small near the horizon. This includes the M2 and M5 branes in eleven dimensions.

V. DISCUSSION

We have shown that there exist a large class of inhomogeneous extended black holes in higher dimension. Unlike the solutions discussed previously [5], these solutions exist even when the translationally invariant black branes are stable. Some of these solutions remain inhomogeneous even when the length of the compact direction at infinity goes to zero. As one consequence, this means that for certain values of the mass and charge, there is more than one stable solution. In other words, the four dimensional black hole uniqueness theorems do not extend to higher dimensions with horizon topology $S^p \times T^q$. It should be noted that even for the vacuum Einstein equation, it has recently been shown that there is no uniqueness theorem. In five dimensions, in addition to the rotating black hole, there is another stationary solution with the same mass and angular momentum which describes a black string in the shape of a large ring [24]. The ring is rotating and the centrifugal force balances the gravitational attraction.

The inhomogeneous solutions we have discussed have higher Bekenstein-Hawking entropy than their homogeneous counterparts. So one might expect that even though the homogeneous solution is classically stable, it might quantum mechanically tunnel to the inhomogeneous state. We cannot yet estimate the probability for this transition, but expect it to be very small for any macroscopic solution.

An important open question is whether one can give a microscopic description of the entropy for these new solutions. For many of the homogeneous near-extremal solutions such a description has been obtained in string theory by taking the limit as the string coupling g goes to zero [9]. The mass and charge remain but no longer gravitate. So the spacetime becomes flat. The charge is represented by D-branes and the excess energy is described by fields on the brane. The number of states turns out to be the exponential of $S_{BH} = A/4$ where A is the area (in Planck units) of the horizon which appears when one increases g. Is there a similar description for the inhomogeneous solutions?

The answer is not yet clear. One obvious problem is that we do not yet have the static inhomogeneous solution explicitly to compute its entropy. However, given the intuition about what the final state looks like, we can begin to construct a microscopic model. Consider the case of near extremal 3-branes connected by a neutral thin 4-brane. We expect that it will settle down to a near extremal 3-brane connected by a smeared 3-brane. At weak coupling, this corresponds to a large number N of 3-branes all localized at the same point on the circle, together with a smaller number N' of 3-branes uniformly spread around the circle. The states of the localized 3-branes are described, as usual, by a $3+1$ dimensional $U(N)$ gauge theory. The states of the smeared 3-brane are most easily counted in weak coupling by applying T-duality [23]. Then it becomes a near extremal 4-brane with nontrivial flat connection, and the low energy excitations are described by a $U(N')$ gauge theory in $4+1$ dimensions. One might hope that the entropy could be reproduced by combining both systems. Of course, at this level, one does not see any sign of inhomogeneity. That must arise from interactions between these states. One can view the existence of these new supergravity solutions as giving a prediction for what happens at strong coupling.

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